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Finitely Determined Members of Varieties of Groups and Rings[1]

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A finitely generated algebra A in a variety $\mathscr{V}$ is called finitely determined in $\mathscr{V}$ if there exists a finite $\mathscr{V}$-consistent set of equalities and inequalities in an alphabet containing the generating set of A, which, together with the identities of $\mathscr{V}$, yields all relations and non-relations of A. Obviously, if the equational theory of $\mathscr{V}$ is recursively enumerable then any finitely determined algebra in $\mathscr{V}$ has solvable word problem. The known algebraic characterizations of groups and semigroups with solvable word problem imply that in the varieties of all groups and all semigroups the members with solvable word problem are finitely determined. We construct a finitely generated center-by-metabelian group with solvable word problem, which is not finitely determined in every group variety $\mathscr{V}$ with $Z\mathfrak{A}^2 \subseteq \mathscr{V} \subseteq \mathfrak{A}^3$. We show that every extension of a finitely generated abelian group by a finite group from a variety $\mathscr{W}$ is finitely determined in every variety $\mathscr{V} \supseteq Z\mathfrak{A}^2\mathscr{W}$. However, in any abelian-by-nilpotent variety no infinite group is finitely determined; moreover, in every variety, in which all finitely presented algebras are residually finite, each finitely determined algebra is finite. In the variety of all associative linear algebras over a finitely generated field every member with solvable word problem is finitely determined. We construct an example, which shows that for the variety of all associative rings it is not true; however, in this variety each torsion-free member with solvable word problem is finitely determined. © 2000 Academic Press

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1. INTRODUCTION

Let $\mathscr{V}$ be a variety of algebras, and Δ be a $\mathscr{V}$-consistent set of equalities and inequalities in $\bar{x}, \bar{y}$ where $\bar{x}$ and $\bar{y}$ are tuples of variables. We say that Δ *determines* $\bar{x}$ in $\mathscr{V}$ if, for any terms t and s in variables $\bar{x}$, either Δ implies $t = s$ in $\mathscr{V}$, or Δ implies $t \neq s$ in $\mathscr{V}$. We call an algebra A *finitely determined*[2] in $\mathscr{V}$ if there exists a finite Δ such that it determines $\bar{x}$, and, for every (equivalently, for some) solution $\bar{a}, \bar{b}$ of Δ in an algebra in $\mathscr{V}$, the tuple $\bar{a}$ generates an isomorphic copy of A. It is easy to show that a finitely generated algebra A is finitely determined in $\mathscr{V}$ iff there are a finitely presented in $\mathscr{V}$ algebra $B \supseteq A$ and $b_1, b_1', \ldots, b_k, b_k'$ in B with $b_i \neq b_i'$ for all i such that any homomorphism of B, which preserves all these inequalities, is injective on A. It follows that every finitely generated subalgebra of a finitely determined in $\mathscr{V}$ algebra is finitely determined in $\mathscr{V}$.

In every variety of finite signature the finite members are obviously finitely determined. The finiteness of signature is essential here: for example, no member of the variety of all algebras of signature $\{c_i : i < \omega\}$, where each c_i is a constant symbol, is finitely determined.

However, in the subvariety of this variety, which is defined by the identities $c_i = c_j$ for all i, j, every finite algebra is finitely determined. So an algebra finitely determined in a variety need not be finitely determined in its overvarieties. Also, an algebra finitely determined in a variety need not be finitely determined in the subvariety generated by this algebra: for example, the infinite cyclic group is finitely determined in the variety of all groups (see Proposition 2.4) but not in the variety of all abelian groups (see Theorem 2.2).

Obviously, if $\mathscr{V}$ has the Joint Embedding Property then every existentially closed algebra in $\mathscr{V}$ embeds each finitely determined member of $\mathscr{V}$. For any variety $\mathscr{V}$ of at most countable signature the converse also holds: A. Macintyre [12] proved in fact that any finitely generated algebra, which is not finitely determined in $\mathscr{V}$, can be omitted in some countable existentially closed algebra in $\mathscr{V}$.

It is easy to see that if the equational theory of $\mathscr{V}$ is recursively enumerable then any finitely determined member of $\mathscr{V}$ has solvable word problem. For the varieties of all groups and all semigroups the converse is also true: in these varieties the members with solvable word problem are finitely determined. This fact is just a version of the known algebraic characterizations of groups and semigroups with solvable word problem [4,

[2] The referee let me know that the term was used by Yu. G. Kleiman in a different sense: he called a *group* finitely determined if it could be defined by finitely many defining relations and identities.

15]. The crucial point in the proofs is a use of the Higman Embedding Theorem [9], which claims that every recursively presented group can be embedded into a finitely presented group, and its semigroup analog due to V. L. Murskiĭ [14].

For the variety of associative rings and the variety of associative linear algebras over any field finitely generated over its prime subfield, analogs of the Higman Embedding Theorem also hold, due to V. Ya. Belyaev [2]. Therefore it seemed plausible that in these varieties every member with solvable word problem is finitely determined. For the variety of associative linear algebras over a field, which is finitely generated over its prime subfield, it is the case indeed (Theorem 3.1). The proof is essentially the same as for the analogous known result for semigroups, because here a similar embedding technique, due to L. A. Bokut' [3], can be applied. However, we construct a finitely generated associative ring with solvable word problem, which is not finitely determined in the variety of all associative rings (Theorem 3.5). Nevertheless, every torsion-free associative ring with solvable word problem is finitely determined in this variety (Theorem 3.3).

We show that in certain solvable group varieties there are groups with solvable word problem, which are not finitely determined in these varieties. Let $\mathfrak{A}^n$ be the variety of all n-step solvable groups, and $Z\mathfrak{A}^2$ be the variety of all center-by-metabelian groups. We construct a finitely generated center-by-metabelian group with solvable word problem, which is not finitely determined in every variety $\mathscr{V}$ such that $Z\mathfrak{A}^2 \subseteq \mathscr{V} \subseteq \mathfrak{A}^3$ (Theorem 2.1). We prove that in every group variety containing $Z\mathfrak{A}^2$ any finitely generated abelian group is finitely determined (Theorem 2.7). Moreover, every extension of a finitely generated abelian group by a finite group from a variety $\mathscr{W}$ is finitely determined in every variety $\mathscr{V} \supseteq Z\mathfrak{A}^2\mathscr{W}$ (Theorem 2.8).

However, in every variety of abelian-by-nilpotent groups no infinite group is finitely determined: we note that in every variety, in which all finitely presented algebras are residually finite, each finitely determined member is finite (Theorem 2.2). So, in case of finite signature, in every such a variety the finitely determined algebras are exactly the finite algebras. Note, however, that there are varieties of infinite signature, in which all algebras are residually finite but no algebra is finitely determined: the simplest example is the variety of all algebras of a signature, which consists of infinitely many constant symbols.

As a special case of the abovementioned result on a connection between finite determinedness and existential closedness, we have that, for any group variety $\mathscr{V}$, the infinite cyclic group is not finitely determined in $\mathscr{V}$ if and only if there exists a periodic existentially closed group in $\mathscr{V}$. Thus, in every variety of abelian-by-nilpotent groups there is a periodic existentially closed group. F. Leinen [11, Question 5.3(a)] posed a problem whether periodic existentially closed groups exist in the varieties $\mathfrak{A}^n$ for $n \geq 3$. Our results give a negative answer to the question: Propositions 2.3 and 2.4 immediately imply that there is no periodic existentially closed group in every variety $\mathscr{V} \supseteq Z\mathfrak{A}^2$.

It seems interesting, for any specific group G in any specific variety $\mathscr{V}$, to decide whether G is finitely determined in $\mathscr{V}$ or not. Some questions of this type are formulated at the end of Section 2. We conclude the introduction with some open questions on finitely determined members of group varieties.

Open Problems. (1) Is a group variety, in which all finitely generated groups with solvable word problem are finitely determined, either the variety of all groups or a locally finite variety?

(2) Characterize the group varieties, in which the finitely generated abelian groups are finitely determined.

(3) Is it true that the direct product of two groups, both of which are finitely determined in a variety, is finitely determined in the variety (cf. Proposition 2.6)?

(4) Is it true that in a group variety every finitely determined member is finite if and only if every group, which is finitely presented in the variety, is residually finite?

(5) Construct a group, which is finitely determined in a group variety, but is not finitely determined in some of its overvarieties.

(6) Is it true that every finitely generated solvable group with solvable word problem, or at least every poly-(finitely generated abelian) group, is finitely determined in the variety $\mathfrak{A}^n$, for some n?

2. FINITELY DETERMINED GROUPS IN SOLVABLE VARIETIES

THEOREM 2.1. *There exists a 2-generated center-by-metabelian group with solvable word problem, which is not finitely determined in every group variety* $\mathscr{V}$ *such that* $Z\mathfrak{A}^2 \subseteq \mathscr{V} \subseteq \mathfrak{A}^3$.

Remark. Finitely generated groups in $Z\mathfrak{A}^2$ with unsolvable word problem do exist (see [1, Theorem 2.4; 13]). However, every finitely presented in $Z\mathfrak{A}^2$ group has solvable word problem [16]. There is a group in $\mathfrak{A}^3$ with unsolvable word problem, which is finitely presented even in the variety of all groups [10].

Proof. Let C be the group defined in the variety of 2-step nilpotent groups by the generators $\{x_i : i \in \mathbb{Z}\}$ and the relations

$$[x_i, x_j] = [x_{i+1}, x_{j+1}] \qquad (i, j \in \mathbb{Z}, i < j).$$

Clearly, the map $x_i \mapsto x_{i+1}$ can be extended to an automorphism φ of C. Let H be the extension of C by φ. Then H is generated by the elements x_0 and φ. Let D be the subgroup of H generated by all $[x_i, x_j]$; clearly, $D \leq Z(H)$. It is easy to see that H/D is isomorphic to the wreath product $\mathbb{Z} \wr \mathbb{Z}$ that is a metabelian group. Thus H is center-by-metabelian. (Note that the construction of central-by-metabelian group with unsolvable word problem in [1, 13] was based on the group H.)

Put $v_i := [x_0, x_k]$, and $u_k := v_{k-1}^{-1} v_k^2 v_{k+1}^{-1}$. We denote by K' the derived subgroup of a group K.

Claim. The subgroup H'' is a free abelian group, and $\{u_k : k > 0\}$ is its basis. Moreover, there is an algorithm which, for any group word w in x_0 and φ, decides whether $w \in H''$ or not, and, in the case of a positive answer, produces a word $u_1^{n_1} \dots u_t^{n_t}$ equal to w in H.

Proof of the Claim. For the proof, we consider the free 2-step nilpotent group F with a basis $\{x_i : i \in \mathbb{Z}\}$. Let c_{ij} denote $[x_i, x_j]$; so c_{0k} is v_k. Group words of the form

$$x_{i_1}^{l_1} x_{i_2}^{l_2} \dots x_{i_n}^{l_n} c_{j_1 k_1}^{m_1} c_{j_2 k_2}^{m_2} \dots c_{j_s k_s}^{m_s},$$

where

- $i_1 < \cdots < i_n$,
- $j_q < k_q$ for all q,
- $(j_1, k_1) < \cdots < (j_s, k_s)$ in the lexicographic order,
- all l_p, m_q are nonzero integers,

will be called normal. (Note that here n or s may be equal to zero.) Every group word w in the x_i's is equal in F to a unique normal word, and this normal word can be found effectively in w (see [7, Chap. 11]). It follows that F' is a central subgroup freely generated by all c_{jk}, $j < k$.

The group C is F/R, where R is the central subgroup of F generated by the elements $c_{ij} c_{i+1, j+1}^{-1}$, $i < j$. Then every group word w in the x_i's is equal in C to a unique normal word of the form

$$x_{i_1}^{l_1} x_{i_2}^{l_2} \dots x_{i_n}^{l_n} c_{0k_1}^{m_1} c_{0k_2}^{m_2} \dots c_{0k_s}^{m_s}, \qquad (*)$$

and this normal word can be found effectively in w. It follows that C' is a central subgroup of H freely generated by $\{v_k : k > 0\}$.

Put $y_i := [x_i, \varphi] = x_i^{-1}x_{i+1}$. Let $A = \langle y_i : i \in \mathbb{Z}\rangle$. We claim that $H' = AC'$. Clearly, $H' \geq AC'$. Let B be the normal closure of A in H. Since the generators φ and x_0 of H commute modulo B, the quotient group H/B is abelian. Hence $H' \leq B$. As $y_i^{\varphi} = y_{i+1}$ and

$$y_i^{x_k} = y_i[y_i, x_k] \in AC',$$

we have $B \leq AC'$.

It follows that the central subgroup H'' is generated by all elements $[y_i, y_j]$, $i < j$. Moreover, if $j - i = k > 0$ then in H

$$\begin{aligned}[y_i, y_j] &= \left[x_i^{-1}, x_j^{-1}\right]\left[x_{i+1}x_j^{-1}\right]\left[x_i^{-1}, x_{j+1}\right][x_{i+1}, x_{j+1}] \\ &= [x_i, x_j][x_{i+1}, x_j]^{-1}[x_i, x_{j+1}]^{-1}[x_{i+1}, x_{j+1}] \\ &= [x_0, x_k][x_0, x_{k-1}]^{-1}[x_0, x_{k+1}]^{-1}[x_0, x_k] \\ &= v_{k-1}^{-1}v_k^2 v_{k+1}^{-1} \\ &= u_k.\end{aligned}$$

We show that $\{u_k : k > 0\}$ is linearly independent over $\mathbb{Z}$. Toward a contradiction, suppose $u_1^{n_1} \dots u_s^{n_s} = 1$ for some integers n_i, and $n_s \neq 0$. Then

$$v_{s+1}^{n_s} = u_1^{n_1} \dots u_{s-1}^{n_{s-1}} v_{s-1}^{-n_s} v_s^{2n_s},$$

and so $v_{s+1}^{n_s} \in \langle v_1, v_2 \dots v_s\rangle$, contrary to linear independence of the v_i's. So the set $\{u_k : k > 0\}$ forms a basis of the free abelian group H''.

Every group word v in φ and x_0 is equal in H to a unique word of the form $\varphi^n w$, where n is an integer, and w is a word of the form $(*)$. Moreover, n and w can be found effectively in v. If at least one of $n, l_1, \dots, l_n$ is nonzero then $v \notin C'$, and so $v \notin H''$. Suppose $n = l_1 = \cdots = l_n = 0$. Then w is a word in the v_i's, and v is equal to w in H. So $v \in C'$. Suppose all these v_i's are among $v_1, \dots, v_t$. It is easy to see that $v \in H''$ iff $w \in \langle u_1, \dots, u_t\rangle$. It is well known that in any free abelian group of finite rank the occurrence problem is solvable; by applying that to the free abelian group $\langle v_1, \dots, v_{t+1}\rangle$ and its subgroup $\langle u_1, \dots, u_t\rangle$, we can decide whether $w \in \langle u_1, \dots, u_t\rangle$ or not, and, in the case of a positive answer, effectively find a word $u_1^{n_1} \dots u_t^{n_t}$ equal to w in H. ∎

Let p_k be the kth prime number. We denote by N the central subgroup of H generated by all $u_k^{p_k}$. Let $G = H/N$. Clearly, G is a 2-generated center-by-metabelian group. The group G has solvable word problem: it easily follows from the claim above and the obvious observation that in H

we have $u_1^{n_1}\dots u_t^{n_t} \in N$ iff p_i divides n_i for each i. So the element $g_k \in G$ represented by the word u_k is of order p_k in G.

Let $\mathscr{V}$ be a group variety such that $Z\mathfrak{A}^2 \subseteq \mathscr{V} \subseteq \mathfrak{A}^3$. We will show that G is not finitely determined in $\mathscr{V}$. It suffices to prove the following

Claim. For every group D in $\mathscr{V}$ with $G \leq D$ and every nonidentity elements $d_1,\dots,d_n \in D$, there is a normal subgroup K of D such that $K \cap G \neq 1$ and $d_1,\dots,d_n \notin K$.

Proof of the Claim. For $k > 0$ and $x \in D$, we have $g_k^x \in D''$ because $g_k \in G''$. Clearly, g_k^x has order p_k. As D is in $\mathscr{V}$, and $\mathscr{V} \subseteq \mathfrak{A}^3$, the subgroup D'' is abelian. Let K_k be the normal closure of g_k in D. We have $g_k \in K_k \cap G$, and hence $K_k \cap G \neq 1$. Since K_k is generated by $\{g_k^x : x \in D\}$, the subgroup K_k is abelian of exponent p_k. Therefore $K_k \cap K_i = 1$ for distinct k and i. Hence there exists k such that $d_1,\dots,d_n \notin K_k$. ∎

The proof of Theorem 2.1 is completed. ∎

Obviously, in every variety of finite signature each of its finite members is finitely determined. We show that for varieties, in which all finitely presented algebras are residually finite, the converse holds as well. We recall that an algebra B is said to be *residually finite* if, for every distinct $b, b' \in B$, there is a homomorphism α from B into a finite algebra with $\alpha(b) \neq \alpha(b')$.

THEOREM 2.2. *Let $\mathscr{V}$ be a variety in which all finitely presented algebras are residually finite. Then every finitely determined algebra in $\mathscr{V}$ is finite.*

Proof. Suppose an algebra A is finitely determined in $\mathscr{V}$. Then there are a finitely presented in $\mathscr{V}$ algebra $B \supseteq A$ and $b_1, b_1', \dots, b_k, b_k'$ in B with $b_i \neq b_i'$ for all i such that any homomorphism of B, which preserves all these inequalities, is injective on A. Let α_i be a homomorphism from B to a finite algebra C_i such that $\alpha_i(b_i) \neq \alpha_i(b_i')$. Then

$$\alpha : x \mapsto (\alpha_1(x),\dots,\alpha_n(x))$$

is a homomorphism from B to the finite algebra $C_1 \times \cdots \times C_n$ such that $\alpha(b_i) \neq \alpha(b_i')$ for all i. Therefore α is injective on A, and so A is finite. ∎

In particular, in group varieties, in which every finitely presented member is residually finite, no infinite group is not finitely determined. Examples of such varieties are arbitrary varieties of abelian-by-nilpotent groups [8]. However, as we shall show later, for sufficiently big varieties there are a lot of infinite groups finitely determined in them.

We shall need the following technical notion. Let A be a finitely generated group in a variety $\mathscr{V}$. We say that A is *strongly finitely determined* in $\mathscr{V}$ if there are a finitely presented group $B \geq A$ and nonidentity elements $b_1, \dots, b_k, c_1, \dots, c_m$ in B such that

(1) there is a homomorphism from B to a member of $\mathscr{V}$, which does not kill $b_1, \dots, b_k$,

(2) if a homomorphism of B does not kill $b_1, \dots, b_k$ then for each nonidentity $a \in A$ there is j with $1 \leq j \leq m$ such that the homomorphism does not kill $[c_j, a]$.

We say in this case that the group B and the sets $\{b_1, \dots, b_k\}, \{c_1, \dots, c_m\}$ witness that A is strongly finitely determined in $\mathscr{V}$.

Clearly, if A is strongly finitely determined in $\mathscr{V}$ then A is strongly finitely determined in every overvariety of $\mathscr{V}$. Any finitely generated subgroup of a group, which is strongly finitely determined in $\mathscr{V}$, is strongly finitely determined in $\mathscr{V}$ as well.

PROPOSITION 2.3. *Any group, which is strongly finitely determined in a variety, is finitely determined in this variety.*

Proof. Suppose a group A is strongly finitely determined in $\mathscr{V}$, and $B \geq A$, b_i, c_j witness that. Let a set of words V define the variety $\mathscr{V}$, and let $V(B)$ be the corresponding verbal subgroup of B. Let β be the canonical epimorphism from B onto $\overline{B} = B/V(B)$; clearly, $\overline{B} \in \mathscr{V}$. By (1) there is a homomorphism α from B to a member C of $\mathscr{V}$, which does not kill $b_1, \dots, b_k$. There exists a homomorphism $\overline{\alpha} : \overline{B} \to C$ with $\alpha = \overline{\alpha} \circ \beta$. Clearly, $\overline{b}_i = \beta(b_i) \neq 1$ for all i. Therefore by (2) for each nonidentity $a \in A$ there is j such that β does not kill $[c_j, a]$ and so does not kill a. Hence β is injective on A. Put $\overline{A} = \beta(A)$; then the subgroup $\overline{A}$ of $\overline{B}$ is an isomorphic copy of A. We claim that $\overline{B}$ and the elements $\overline{b}_i$ witness that $\overline{A}$ is finitely determined in V. Indeed, if γ is a homomorphism of $\overline{B}$, which does not kill all of the elements $\overline{b}_i$, then $\gamma \circ \beta$ is a homomorphism of B, which does not kill all of the elements b_i; hence by (2) for each nonidentity $a \in A$ there is j such that $\gamma \circ \beta$ does not kill $[c_j, a]$ and so does not kill a. Therefore $\gamma \circ \beta$ is injective on A, and so γ is injective on $\overline{A}$. ∎

Remark. A group, which is finitely determined in a variety, need not be strongly finitely determined in the variety. For example, no nonabelian group is strongly finitely determined in the variety of 2-step nilpotent groups. Indeed, suppose A is a strongly finitely determined nonabelian member of the variety, and $B \geq A$, b_i, c_j witness that. Let a be a nonidentity commutator in A. Then, in notation of Proposition 2.3, $\bar{a}$ is a nonidentity element of $Z(\overline{B})$, and so $[\bar{c}_j, \bar{a}] = 1$ for all j, a contradiction. So every finite nonabelian 2-step nilpotent group is finitely determined but not strongly finitely determined in the variety.

PROPOSITION 2.4. *The infinite cyclic group is strongly finitely determined in the variety of all center-by-metabelian groups.*

Proof. Let B be the group defined in the variety of all groups by the presentation

$$\langle u, v, w; v^{-1}uv = u^2, u^2w = wu^2 \rangle.$$

The group B can be constructed as follows: take the HNN-extension

$$\langle u, v; v^{-1}uv = u^2 \rangle$$

of the infinite cyclic group generated by u, and then amalgamate this HNN-extension with the direct product of two infinite cyclic groups generated by z and w, by the relation $u^2 = z$. It follows that in B the element u is of infinite order, and $[u, w] \neq 1$. Let $A = \langle u \rangle$. Put $b := [u, w]$ and $c := v$. We will show that B, b, c witness that A is strongly finitely determined in $Z\mathfrak{A}^2$.

First we show that if elements u, v, w of an arbitrary group satisfy

$$v^{-1}uv = u^2, \qquad u^2w = wu^2, \qquad uw \neq wu \qquad (*)$$

then u is of infinite order, and, moreover, $[v, u^n] \neq 1$ for each nonzero n. Toward a contradiction, suppose the order of u is finite. If the order of u is an even number $2n$ then the order of u^2 is n; hence u and u^2 are not conjugate, contrary to the first of the equations. If the order of u is odd then the elements u and u^2 are of the same order, and so u^2 generates $\langle u \rangle$. Then w commutes with u and if and only if w commutes with u^2, a contradiction. Now if $[v, u^n] = 1$ then $u^n = v^{-1}u^n v = u^{2n}$ and so $n = 0$.

Thus (2) holds; it suffices to prove (1). That is, the system $(*)$ has a solution in some group G in $Z\mathfrak{A}^2$.

Now we construct a group G with the desired properties.

We shall need the following construction. Let A, B, C be additive abelian groups, and $g: A \times B \to C$ be a bilinear map. Define a map $f: (A \times B)^2 \to C$ by $f((a, b), (a', b')) = g(a, b')$. Then f is a normalized 2-cocycle of the group $A \times B$ with coefficients in C, and so

$$(a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c' + g(a, b'))$$

is a group operation on the set $A \times B \times C$ (see [5, Sect. 3, Chap. IV]). Let $[A, B, C, g]$ denote the group just defined. We can identify C with the subgroup $\{(0, 0, c): c \in C\}$ of this group. Clearly, the group $[A, B, C, g]$ is

2-step nilpotent: it is an extension of C by $A \times B$, where the action of $A \times B$ on C is trivial.

Let A be the additive abelian group freely generated by $\{x_n : n \in \mathbb{Z}\}$.

Let B be the additive group defined in the variety of abelian groups by the generators $\{y_n : n \in \mathbb{Z}\}$ and the relations $\{2y_{n+1} = y_n : n \in \mathbb{Z}\}$. It is isomorphic to $\mathbb{Q}_2$, the additive group of rational numbers with denominators, which are powers of two.

Let C be the additive group defined in the variety of abelian groups by the generators $\{z_n : n \in \mathbb{Z}\}$ and the relations

$$\{2z_{n+1} = z_n : n \in \mathbb{Z}\} \cup \{z_n = 0 : n \leq 0\}.$$

So C is the additive Prüfer 2-group; it is isomorphic to $\mathbb{C}(2^\infty)$, the multiplicative group of complex roots of unity of degrees of the form 2^n, $n > 0$.

For every integer n the map $y_k \mapsto z_{k-n}$ can be extended to a homomorphism $\pi_n : B \to C$. The map $x_n \mapsto \pi_n$ can be extended to a homomorphism of A to the group $\mathrm{Hom}(B, C)$. Let $\hat{a}$ denote the image of an element a under this homomorphism; so $\hat{x}_n = \pi_n$. Clearly, the map $g : (a, b) \mapsto \hat{a}(b)$ is a bilinear map from $A \times B$ to C. Put $H := [A, B, C, g]$.

Let $b \in B$. For every integers n and k,

$$\pi_n(y_k) = z_{k-n} = 2z_{k-n+1} = \pi_{n-1}(2y_k).$$

Therefore, for every integer n, we have $\pi_n(b) = \pi_{n-1}(2b)$, or, in other words, $g(x_n, b) = g(x_{n-1}, 2b)$. Let τ be an automorphism of A, which takes x_n to x_{n-1} for each n. So we have $g(x_n, b) = g(\tau(x_n), 2b)$ for every integer n, and then $g(a, b) = g(\tau(a), 2b)$ for every $a \in A$.

Consider the map $\psi : H \to H$ defined by $\psi(a, b, c) = (\tau(a), 2b, c)$. Clearly, ψ is a bijection. Moreover, ψ is an automorphism of the group H,

$$\begin{aligned}
&\psi((a, b, c) \cdot (a', b', c')) \\
&\quad = \psi(a + a', b + b', c + c' + g(a, b')) \\
&\quad = (\tau(a + a'), 2(b + b'), c + c' + g(a, b')) \\
&\quad = (\tau(a) + \tau(a'), 2b + 2b', c + c' + g(\tau(a), 2b') \\
&\quad = (\tau(a), 2b, c) \cdot (\tau(a'), 2b', c') \\
&\quad = \psi(a, b, c) \cdot \psi(a', b', c').
\end{aligned}$$

Let G be the extension of the group H by the automorphism ψ. As ψ is the identity on C, and $C \leq Z(H)$, we have $C \leq Z(G)$. Since

$$H/C \simeq A \times B \qquad \text{and} \qquad (G/C)/(H/C) \simeq G/H \simeq \mathbb{Z},$$

the group G/C is metabelian. So G is center-by-metabelian.

Put $u := (0, y_1, 0)$, $v := \psi$, $w := (x_0, 0, 0)$. Then u, v, w satisfies the system $(*)$ in G. Indeed, first,

$$v^{-1}uv = \psi(0, y_1, 0) = (\tau(0), 2y_1, 0) = (0, 2y_1, 0) = u^2.$$

Second,

$$\begin{aligned} wu^2 &= (x_0, 0, 0) \cdot (0, 2y_1, 0) = (x_0, 2y_1, \pi_0(2y_1)) = (x_0, 2y_1, 2z_1) \\ &= (x_0, 2y_1, 0), \\ u^2 w &= (0, 2y_1, 0) \cdot (x_0, 0, 0) = (x_0, 2y_1, 0). \end{aligned}$$

Third,

$$\begin{aligned} wu &= (x_0, 0, 0) \cdot (0, y_1, 0) = (x_0, y_1, \pi_0(y_1)) = (x_0, y_1, z_1), \\ uw &= (0, y_1, 0) \cdot (x_0, 0, 0) = (x_0, y_1, 0). \end{aligned}$$

Since $z_1 \neq 0$ in the group C, we have $uw \neq wu$.

Proposition 2.4 is proved. ∎

PROPOSITION 2.5. *Any finite abelian group is strongly finitely determined in the variety of all 2-step nilpotent groups.*

Proof. For a finite abelian group A, let $B = A * C$, where C is an infinite cyclic group generated by c. Clearly, $A \leq B$, and the group B is finitely presented. Let $A \backslash \{1\} = \{a_1, \dots, a_k\}$. Put $b_i := [c, a_i]$. We claim that $B, b_1, \dots, b_k, c$ witness that A is strongly finitely determined in the variety of all 2-step nilpotent groups.

It suffices to show that there exist a 2-step nilpotent group $D \geq A$ and $d \in D$ such that $[d, a] \neq 1$ for all $a \in A \backslash \{1\}$. In fact, this holds for an arbitrary, not necessarily finite, abelian group A. To construct such a group, consider a bilinear map $g : (a, n) \mapsto na$ from $A \times \mathbb{Z}$ to A; here we assume that A is an additive group. Consider the 2-step nilpotent group $D = [A, \mathbb{Z}, A, g]$, in notation from the proof of Proposition 2.4. Put $d := (0, 1, 0)$. The map $a \mapsto \hat{a} = (a, 0, 0)$ is an embedding of A into D. As for any nonzero $a \in A$

$$\hat{a}d = (a, 1, a) \neq (a, 1, 0) = d\hat{a},$$

the result follows. ∎

PROPOSITION 2.6. *The direct product of finitely many groups, all of which are strongly finitely determined in a variety, is strongly finitely determined in the variety.*

Proof. Clearly, it suffices to prove the result for two groups. Let the groups A and A' be strongly finitely determined in a variety $\mathscr{V}$. Let

B, b_i, c_j and B', b'_i, c'_j witness that. We consider B and B' as subgroups of $B \times B'$. Clearly, $B \times B'$ is finitely presented, and $B \times B' \geq A \times A'$. We claim that $B \times B'$, b_i, b'_i, c_j, c'_j witness that $A \times A'$ is strongly finitely determined in $\mathscr{V}$. If α is a homomorphism from B to a member of $\mathscr{V}$, which does not kill all of the b_i, and α' is a homomorphism from B' to a member of $\mathscr{V}$, which does not kill all of the b'_i, then $\alpha \times \alpha'$ is a homomorphism from $B \times B'$ to a member of $\mathscr{V}$, which does not kill all of the b_i, b'_i. Let β be a homomorphism from $B \times B'$ to a member of $\mathscr{V}$, which does not kill all of the b_i, b'_i. Consider a nonidentity element aa' in $A \times A'$, where $a \in A$ and $a' \in A'$; suppose, say, $a \neq 1$. There exists j such that β does not kill $[c_j, a]$. As $[c_j, A'] = 1$, we have $[cj, aa'] = [c_j, a]$, and the result follows. ∎

THEOREM 2.7. *Every finitely generated abelian group is strongly finitely determined in $Z\mathfrak{A}^2$, and, in particular, finitely determined in any variety $\mathscr{V} \supseteq Z\mathfrak{A}^2$.*

Proof. Since every finitely generated abelian group is a direct product of finitely many cyclic groups, the result immediately follows from Propositions 2.3, 2.4, 2.5, 2.6. ∎

THEOREM 2.8. *Let a group A be strongly finitely determined in a variety $\mathscr{V}$, and G be a finite group in a variety $\mathscr{W}$. Then every group extension of A by G is finitely determined in each variety $\mathscr{U} \supseteq \mathscr{V}\mathscr{W}$.*

Proof. Since every extension of A by G is embeddable into the wreath product $A \wr G$ (see [6, Theorem 2.6A]), it suffices to prove that $A \wr G$ is finitely determined in each variety $\mathscr{U} \supseteq \mathscr{V}\mathscr{W}$.

If $A = 1$ then the result trivially holds; so we assume $A \neq 1$. Let $B, b_1, \dots, b_k, c_1, \dots, c_m$ witness that A is strongly finitely determined in $\mathscr{V}$. Denote $B \wr G$ by C, and $A \wr G$ by D. Clearly, $D \leq C$. We consider G and B^G as subgroups of C, and identify B with its "first" copy in the direct power B^G. Then the "gth" copy of B in B^G is the conjugate B^g of B in C. Since G is finite and B is finitely presented, C is finitely presented as well.

We show that every homomorphism α of C, which does not kill the elements b_i, is injective on D.

First we prove that α is injective on the direct power A^G. Let $v \in A^G$; then v is a product of pairwise commuting elements of the form a_g^g, where $a_g \in A$ and $g \in G$. Suppose $v \neq 1$; then $a_h \neq 1$ for some $h \in G$. Choose j such that α does not kill $[c_j, a_h]$. As $[B^h, B^g] = 1$ for $g \neq h$, we have $[c_j^h, v] = [c_j, a_h]^h$; so α does not kill $[c_j^h, v]$ and so does not kill v.

Since D is a semidirect product of A^G and G, it remains to show that if $\alpha(g) = \alpha(v)$ for $g \in G$ and $v \in A^G$ then $g = 1$. For $a \in A \backslash \{1\}$, we have $\alpha(a^v) = \alpha(a^g)$. If $g \neq 1$ then $A \cap A^g = 1$, and hence $\alpha(A) \cap \alpha(A^g) = 1$

because of injectivity of α on A^G. Since $a^v \in A$ and $a^g \in A^g$, it follows that $\alpha(a^g) = 1$, and, using injectivity of α on A^G again, we have $a^g = 1$ and hence $a = 1$, a contradiction.

Let a set of words V define the variety $\mathscr{V}$, and let $V(B)$ be the corresponding verbal subgroup of B. Let β be the canonical epimorphism from B onto $\overline{B} = B/V(B)$; then $\overline{B} \in \mathscr{V}$. As in the proof of Proposition 2.3, β does not kill the elements b_i. The epimorphism β induces an epimorphism γ from C onto $\overline{B} \wr G$; clearly, γ does not kill the elements b_i.

Let a set of words U define the variety $\mathscr{U}$, and let $U(C)$ be the corresponding verbal subgroup of C. Let δ be the canonical epimorphism from C onto $\tilde{C} = C/U(C)$; then the group $\tilde{C}$ is finitely presented in $\mathscr{U}$. Clearly, the group $\overline{B} \wr G$ is in $\mathscr{V}\mathscr{W}$ and so in $\mathscr{U}$. There exists a homomorphism $\zeta : \tilde{C} \to \overline{B} \wr G$ with $\gamma = \zeta \circ \delta$. Hence δ does not kill the elements b_i and so is injective on D. So the subgroup $\tilde{D} = \delta(D)$ of $\tilde{C}$ is an isomorphic copy of D.

The group $\tilde{C}$ and the elements $\delta(b_i)$ witness that $\tilde{D}$ is finitely determined in $\mathscr{U}$. Indeed, suppose a homomorphism ρ of $\tilde{C}$ does not kill the elements $\delta(b_i)$. Then the homomorphism $\rho \circ \delta$ of C does not kill the elements b_i; hence it is injective on D. Therefore ρ is injective on $\tilde{D}$. ∎

THEOREM 2.9. *Let A be a finitely generated abelian group, and G be a finite group in a variety $\mathscr{W}$. Then every group extension of A by G is finitely determined in each variety $\mathscr{U} \supseteq Z\mathfrak{A}^2\mathscr{W}$.*

Proof. As A is strongly finitely determined in $Z\mathfrak{A}^2$ by Proposition 2.7, we may apply Theorem 2.8. ∎

For example, the metabelian group $\mathbb{Z} \wr \mathbb{Z}_n$ is finitely determined in $\mathfrak{A}^4$. Because of Theorem 2.2, this group is not finitely determined in $\mathfrak{A}^2$; however, I do not know whether it is finitely determined in $\mathfrak{A}^3$.

The metabelian group $\mathbb{Z} \wr \mathbb{Z}$ has solvable word problem and so is finitely determined in the variety of all groups, but I do not know whether there exists n such that this group is finitely determined in $\mathfrak{A}^n$. Again, because of Theorem 2.2, it is certainly not finitely determined in $\mathfrak{A}^2$.

3. FINITELY DETERMINED ASSOCIATIVE RINGS AND ALGEBRAS

For a field F we consider linear F-algebras as algebras of signature

$$\{+, \cdot, -, 0, f_\alpha\}_{\alpha \in F},$$

where the unary operation f_α is interpreted as the multiplication by α. Clearly, the F-algebras forms a variety; if the field F is recursive then the equational theory of this variety is recursively axiomatizable. In particular, it is the case if F is a field finitely generated over its prime subfield.

THEOREM 3.1. *Let F be a field finitely generated over its prime subfield. Then the finitely determined associative F-algebras are exactly the finitely generated associative F-algebras with solvable word problem.*

Proof. We shall use the following known fact [3].

Fact 3.2. For any associative algebra A over a field F and any $a, b \in A$ with $a \neq 0$ there are an associative F-algebra $B \supseteq A$ and $u, v \in B$ such that $uav = b$.

Let A be an F-algebra with solvable word problem, generated by a tuple $\bar{x}$. Let T be the set of all terms of the language of F-algebras with variables $\bar{x}$. Let $t_0, t_1, \ldots$ be an effective sequence of all terms in T, which are not equal to 0 in A, and $s_0, s_1, \ldots$ be an effective sequence of all terms in T, which are equal to 0 in A. Consider the recursively presented associative F-algebra D defined by the generators $\bar{x}, u_n, v_n$ and the relations $s_n = 0$, $u_n t_n v_n = t_0$, for all $n < \omega$. The algebra D naturally embeds A, because by iterated using of the fact above we can find an algebra $C \supseteq A$ and $u_n, v_n \in C$ for $n < \omega$, such that $u_n t_n v_n = t_0$ for all n. By Belyaev's theorem [2], D can be embedded into a finitely presented associative F-algebra B. Obviously, each homomorphism of B, which does not kill t_0, is injective on A. Therefore A is finitely determined in the variety of all associative F-algebras. ∎

THEOREM 3.3. *A finitely generated torsion-free associative ring is finitely determined in the variety of all associative rings iff it has solvable word problem.*

Proof. We need the following corollary of Fact 3.2.

LEMMA 3.4. *Let A be a torsion-free associative ring. Then for any $a, b \in A$ with $a \neq 0$ there are a torsion-free associative ring $B \supseteq A$ and $u, v \in B$ such that $uav = b$.*

Proof of Lemma. The map $a \mapsto a \otimes 1$ embeds A into the associative ring $A \otimes_{\mathbb{Z}} \mathbb{Q}$, on which a structure of $\mathbb{Q}$-algebra can be naturally defined. We shall identify a with $a \otimes 1$. By the fact above, there exists an associative $\mathbb{Q}$-algebra $B \supseteq A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $u, v \in B$ such that $uav = b$. Clearly, the ring B is torsion-free. ∎

Now, using Lemma 3.4, we can proceed exactly as in the proof of Theorem 3.1; the only difference is that here we use Belyaev's theorem [2] on embedding of a recursively presented associative ring into a finitely presented associative ring. ∎

Theorem 3.5 shows that the condition "A is torsion-free" in Theorem 3.3 is essential. The reason why the analog of Lemma 3.4 for the class of arbitrary associative rings fails is that there is an obvious necessary condition for solvability of the equation $uav = b$ over A: for any integer n, if $na = 0$ then $nb = 0$. The idea of the example below is based on this observation.

THEOREM 3.5. *There exists a finitely generated associative ring with solvable word problem, which is not finitely determined in the variety of associative rings.*

Proof. Let R be the ring, which is defined in the variety of all associative rings by the generators x, y, z and the relations $pxy^pz = 0$, where p runs over the prime numbers. We will prove that R has the desired properties. First we prove that R has solvable word problem.

Let f be a polynomial with integral coefficients in noncommuting variables x, y, z. We call f *reduced* if

(a) for every prime p, the coefficient of each of its monomials, which has a subword of the form xy^pz, belongs to $\{0, 1, \dots, p-1\}$;

(b) none of its monomials has subwords of the forms xy^pz and xy^qz, for some distinct primes p and q.

Claim. For every polynomial f with integral coefficients in noncommuting variables x, y, z there is a unique reduced polynomial $\bar{f}$ such that f and $\bar{f}$ are equal in R; this $\bar{f}$ can be found effectively in f.

Proof of the Claim. In order to find $\bar{f}$, delete all of the monomials of f, which have subwords of the forms xy^pz and xy^qz, for some distinct primes p and q. For each of the remaining monomials, which has a subword of the form xy^pz, replace its coefficient with the remainder of the coefficient modulo p. Clearly, the constructed polynomial $\bar{f}$ is reduced. Note that if a monomial g has a subword of the form xy^pz for a prime p then $pg = 0$ in R. Therefore every monomial deleted at the first step of the reduction is equal to 0 in R, and at the second step of the reduction we replace monomials with monomials equal to them in R. Thus f is equal to $\bar{f}$ in R.

It remains to show that if reduced polynomials f and h are equal in R then they are equal in F, the ring of all polynomials with integral coefficients in noncommuting variables x, y, z. Suppose $f - h$ belongs to the ideal of F generated by the set $\{pxy^pz : p \text{ is prime}\}$. In other words,

$f-g$ is equal in F to

$$\sum_{p,v,w} n_{p,v,w}pvxy^p zw, \tag{†}$$

where p is prime, and v, w are words in x, y, z, and only finitely many of the integral coefficients $n_{p,v,w}$ are nonzero. Let I be the set of all triples (p, v, w), for which $n_{p,v,w}$ is nonzero, and the monomial $vxy^p zw$ has a subword of the form $xy^q z$ for some prime q with $q \neq p$. Since the set of all monomials is linearly independent over $\mathbb{Z}$ in F, and f and h are reduced, we have in F

$$\sum_{(p,v,w)\in I} n_{p,v,w}pvxy^p zw = 0.$$

Therefore we may assume that in (†) for every (p, u, v) the monomial $vxy^p zw$ has no subword of the form $xy^q z$ with a prime $q \neq p$. Let P be the set of all primes involved in (†). For $p \in P$ let M_p denote the set of all monomials of the form $vxy^p zw$ in the sum (†). Then $M_p \cap M_q = \varnothing$ for $p \neq q$. Then (†) can be written as

$$\sum_{p\in P} \sum_{u\in M_p} k_{p,u}pu,$$

where for $u \in M_p$ the integer $k_{p,u}$ is the sum of all $n_{p,v,w}$ with $vxy^p w = u$. Let $p \in P$ and $u \in M_p$. If m and l are the integral coefficients of the monomial u in f and h, respectively, then $m - l = k_{p,u}p$ because of linear independence over $\mathbb{Z}$ of the set of all monomials in F. Since $0 \leq m, l < p$, we have $m = l$. Thus $f = h$ in F. ▮

It follows from the claim that the word problem is solvable for R: a polynomial f is equal to 0 in R iff $\bar{f}$ is the zero polynomial.

Now we prove that the ring R is not finitely determined in the variety of all associative rings. It suffices to prove the following

Claim. For every associative ring D with $R \leq D$ and nonzero elements $d_1, \ldots, d_n$ in D, there is an ideal K of D such that $d_1, \ldots, d_n \notin K$ and $K \cap R \neq 0$.

Proof of the Claim. For any prime p, the nonzero polynomial $xy^p z$ is reduced and so represents a nonzero element in R. Denote by K_p the ideal of D generated by the element $xy^p z$. The nonzero element $xy^p z$ is in $K_p \cap R$; hence $K_p \cap R \neq 0$. As $pxy^p z = 0$ in R, we have $pK_p = 0$, and so $K_p \cap K_q = 0$ for distinct p and q. Since there are infinitely many primes, there exists p such that $d_1, \ldots, d_n \notin K_p$. ▮

The proof of Theorem 3.5 is completed. ▮

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